1st order PDE

$u(x, y)$: unknown

$$\begin{cases} F(x, y, u, u_x, u_y) = 0 \\ u(\gamma(t)) = \phi(t) : \text{prescribed initial data} \end{cases}$$

If the initial data is **non-characteristic**, then there is the unique solution of the differential equation.

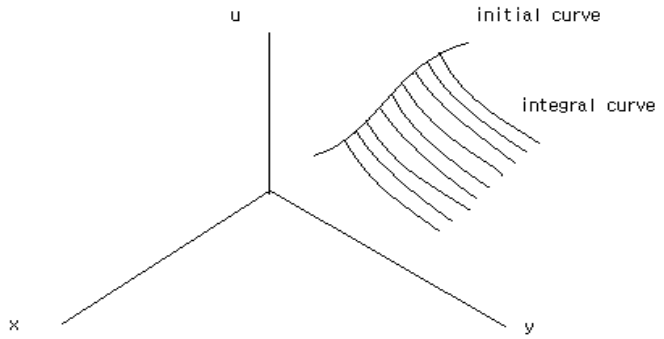
0.1 Quasi-linear case

$$\begin{cases} a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \\ u(x(t), y(t)) = \phi(t) \end{cases}$$

Consider a vector field in $\mathbb{R}^3 = \{(x, y, u)\}$

$$\xi = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial u} : \text{characteristic vector field}$$

and a space curve $(x(t), y(t), \phi(t))$.



Considering integral curves of ξ starting from the initial curve, we get a surface $u = u(x, y)$. Then $u = u(x, y)$ satisfies the initial condition trivially and $a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) = (a(x, y, u), b(x, y, u), c(x, y, u)) \cdot (u_x, u_y, -1) = 0$ because ξ is tangent to the surface $u = u(x, y)$.

0.2 General case

We use a similar argument to the quasi-linear case, in $\{(x, y, u, p, q)\}$ space
$\begin{cases} F(x, y, u, p, q) = 0 \\ u(x(t), y(t)) = \phi(t) \end{cases}$

F gives 1-parameter family $(p(\lambda), q(\lambda))$ of tangent spaces of possible solutions $u = u(x, y)$ at each point (x_0, y_0, u_0) which defines Monge cone with vertex (x_0, y_0, u_0) .

If $u = u(x, y)$ is a solution, then at each point $(x_0, y_0, u(x_0, y_0))$, the tangent space is tangent to the Monge cone. So there is the unique generating line of the cone that is tangent to the solution surface.

In fact, given 1-jet $(x_0, y_0, u_0, p_0, q_0)$ in $F = 0$

$$\begin{cases} F(x, y, u, p(\lambda), q(\lambda)) &= 0 \\ du - p(\lambda)dx - q(\lambda)dy &= 0 \\ p'(\lambda)dx + q'(\lambda)dy &= 0 \\ F_p p'(\lambda) + F_q q'(\lambda) &= 0 \end{cases}$$

Hence, along generating line

$$dx : dy : du = F_p : F_q : pF_p + qF_q \qquad \text{characteristic direction}$$

Thus, at (x_0, y_0, u_0) we have 1-parameter family of characteristic directions.
In quasi-linear case, only one characteristic direction$(F_p : F_q : pF_p + qF_q = a : b : c$)

A curve $(x(s), y(s), u(s))$ is called a focal curve(Monge curve) if

$$(*) \begin{cases} \frac{dx}{ds} &= F_p \\ \frac{dy}{ds} &= F_q \\ \frac{du}{ds} &= pF_p + qF_q \end{cases}$$

A curve $(x(s), y(s), u(s), p(s), q(s))$ in 1st jet space $\{(x, y, u, p, q)\}$ that satisfies (*) and $F(x(s), y(s), u(s), p(s), q(s)) = 0$ is called a focal strip.

For a focal curve to be embedded into an solution surface, we need two additional conditions.

By differentiating $F(x, y, u, p, q) = 0$ with respect to x and y, we get

$$\begin{cases} F_x + F_u p + F_p p_x + F_q q_x &= 0 \\ F_y + F_u q + F_p p_y + F_q q_y &= 0 \end{cases}$$

$$\text{Since} \begin{cases} F_p p_x + F_q q_x &=& \frac{dx}{ds} p_x + \frac{dx}{ds} p_y &=& \frac{dp}{ds} \\ F_p p_y + F_q q_y &=& \frac{dx}{ds} q_x + \frac{dy}{ds} p_y &=& \frac{dq}{ds} \end{cases}$$

Hence,

$$\begin{array}{rcl} \frac{dp}{ds} &=& -(F_x + F_u p) \\ \frac{dq}{ds} &=& -(F_y + F_u p) \end{array}$$

Thus, we get the equation for a focal curve embedded in the solution surface.

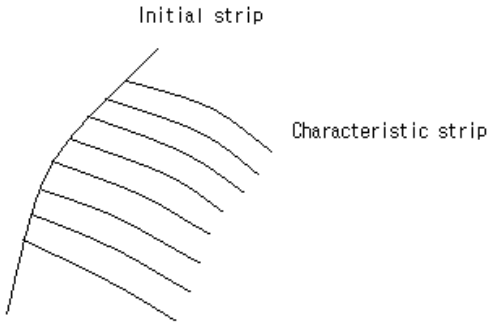
$$\begin{cases} \frac{dx}{ds} &=& F_p \\ \frac{dy}{ds} &=& F_q \\ \frac{du}{ds} &=& pF_p + qF_q \\ \frac{dp}{ds} &=& -(F_x + F_u p) \\ \frac{dq}{ds} &=& -(F_y + F_u p) \end{cases}$$

Such curve is called a characteristic strip.

A curve $(x(t), y(t), u(t), p(t), q(t))$ in 1st jet space is called a initial strip if

$$\begin{cases} du - pdx - qdy &=& 0 \\ F(x(t), y(t), u(t), p(t), q(t)) &=& 0 \end{cases}$$

In $J^1(\mathbb{R}^2, \mathbb{R}) = \{(x, y, u, p, q)\}$



we get 2-dimensional surface $(x(t, s), y(t, s), u(t, s), p(t, s), q(t, s))$
So, $u = u(x, y)$ is obtained.